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# A survey on Shapovalov determinants of (generalized) quantum groups at roots of 1

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## Abstract

This is an informal survey on a joint work [HY08b] with Istvan Heckenberger.

## 1 A quantum group $U(\chi)$ defined for any bi-character $\chi$

Recently study of Nichols algebras has been achieved very actively for the viewpoint of classification of Hopf algebras, see [AS98], [AS02], [Hec06]. One of their examples is the positive part  $U^+(\chi)$  of a generalized quantum group  $U(\chi)$  defined below.

Let  $\mathbb{k}$  be a field and  $\mathbb{k}^\times = \mathbb{k} \setminus \{0\}$ . For  $n \in \mathbb{Z}_{\geq 0}$  and  $x \in \mathbb{k}$ , let

$$(1) \quad [n]_x = \sum_{m=1}^n x^{m-1}, \quad [n]_x! = \prod_{m=1}^n [m]_x.$$

For two elements  $X_1$  and  $X_2$  of a  $\mathbb{k}$ -algebra we use the convention:

$$(2) \quad X_1 \quad X_2 \quad \begin{array}{c} \Longleftrightarrow \\ \text{def} \end{array} \quad \exists x \in \mathbb{k}^\times \quad X_1 = xX_2.$$

Let  $I$  be a finite index set. Let  $\mathbb{Z}\Pi = \sum_{i \in I} \mathbb{Z}\alpha_i$  be a rank  $|I|$  free  $\mathbb{Z}$ -module with a basis  $\Pi = \{\alpha_i | i \in I\}$ . We say that a map  $\chi : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \rightarrow \mathbb{k}^\times$  is a *bi-character* if  $\chi(a+b, c) = \chi(a, c)\chi(b, c)$ , and  $\chi(a, b+c) = \chi(a, b)\chi(a, c)$  for all  $a, b, c \in \mathbb{Z}\Pi$ .

Let  $\chi$  be *any* bi-character. Then, as we explain more precisely in Section 2, Lusztig's definition [L, 3.1.1] of the quantum groups can be applied to define the Hopf  $\mathbb{k}$ -algebra  $U(\chi)$  with the generators

$$(3) \quad K, L \quad (\lambda \in \mathbb{Z}\Pi), E_i, F_i \quad (i \in I),$$

for which  $K, L \in \mathbb{Z}\Pi$  ( $\lambda, \mu \in \mathbb{Z}\Pi$ ) are linearly independent and the following equations hold:

- (4)  $K_0 = L_0 = 1, \quad K_+ = K K, \quad L_+ = L L, \quad K L = L K,$
- (5)  $K L E_j (K L)^{-1} = \frac{\chi(\lambda, \alpha_j)}{\chi(\alpha_j, \mu)} E_j, \quad K L F_j (K L)^{-1} = \frac{\chi(\alpha_j, \mu)}{\chi(\lambda, \alpha_j)} F_j,$
- (6)  $E_i F_j - F_j E_i = \delta_{ij} (K_i - L_i).$
- (7)  $\Delta(K L) = K L - K L, \quad \varepsilon(K L) = 1, \quad S(K L) = (K L)^{-1},$
- (8)  $\Delta(E_i) = E_i - 1 + K_i - E_i, \quad \Delta(F_i) = F_i - L_i + 1 - F_i,$
- (9)  $\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i L_i^{-1}.$

Let  $U^0(\chi) := \sum_{\lambda, \mu \in \mathbb{Z}\Pi} \mathbb{k} K L$ . Let  $U^+(\chi)$  and  $U^-(\chi)$  be the subalgebra of  $U(\chi)$  generated by  $E_i$  and  $F_i$  with all  $i \in I$  respectively. Then  $U(\chi) = U^+(\chi) U^0(\chi) U^-(\chi)$ , as a  $\mathbb{k}$ -linear space. We have the  $\mathbb{Z}_{\geq 0}\Pi$ -grading  $U^\pm(\chi) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}\Pi} U^\pm(\chi)_\lambda$  defined by  $U^+(\chi)_\lambda = \mathbb{k} E_i$ ,  $U^-(\chi)_\lambda = \mathbb{k} F_i$ , and  $U^\pm(\chi) U^\pm(\chi) \subset U^\pm(\chi)_{\lambda + \mu}$ . We also have  $\dim U^-(\chi)_\lambda = \dim U^+(\chi)_\lambda$  for all  $\lambda \in \mathbb{Z}_{\geq 0}\Pi$ .

## 2 Drinfeld pairing of $U(\chi)$

Here we will explain how to define  $U(\chi)$  more precisely. By abuse of notation, we use the same symbols as above for the generators of the algebras introduced in this paragraph. Let  $\tilde{U}^+(\chi)$  and  $\tilde{U}^-(\chi)$  be the free  $\mathbb{k}$ -algebras (with 1) with the generators  $\{E_i | i \in I\}$  and  $\{F_i | i \in I\}$  respectively. Let  $\tilde{U}^0(\chi)$  be the  $\mathbb{k}$ -linear space with the basis  $\{K L | \lambda, \mu \in \mathbb{Z}\Pi\}$ . Let  $\tilde{U}(\chi) = \tilde{U}^+(\chi) \mathbb{k} \tilde{U}^0(\chi) \mathbb{k} \tilde{U}^-(\chi)$ . Identify  $X \in \tilde{U}^+(\chi)$ ,  $Z \in \tilde{U}^0(\chi)$  and  $Y \in \tilde{U}^-(\chi)$  with  $X - 1 - 1, 1 - Z - 1$  and  $1 - 1 - Y$  respectively, and regard  $\tilde{U}^+(\chi)$ ,  $\tilde{U}^0(\chi)$  and  $\tilde{U}^-(\chi)$  as subspaces of  $\tilde{U}(\chi)$  in this way. Then  $\tilde{U}(\chi)$  can be regarded as the  $\mathbb{k}$ -algebra (with 1) presented by the same generators as the ones for  $U(\chi)$  and the relations (4), (5) and (6) (cf. [L, Prop. 3.2.4]). Further  $\tilde{U}(\chi)$  can be regarded as the Hopf  $\mathbb{k}$ -algebra with the same equalities as (7), (8) and (9). Let  $\tilde{U}^{+,K}(\chi)$  be the subalgebra of  $\tilde{U}(\chi)$  generated by  $E_i$ 's and  $K$ 's. Let  $\tilde{U}^{L,-}(\chi)$  be the subalgebra of  $\tilde{U}(\chi)$  generated by  $F_i$ 's and  $L$ 's. Then there exists a unique  $\mathbb{k}$ -bilinear form

$$(10) \quad \langle \cdot, \cdot \rangle : \tilde{U}^{+,K}(\chi) \times \tilde{U}^{L,-}(\chi) \rightarrow \mathbb{k}$$

with

$$(11) \quad \langle 1, Y \rangle = \varepsilon(Y), \quad \langle X, 1 \rangle = \varepsilon(X), \quad \langle S(X), Y \rangle = \langle X, S^{-1}(Y) \rangle,$$

$$(12) \quad \langle X_1 X_2, Y \rangle = \sum_g \langle X_2, Y_g^{(1)} \rangle \langle X_1, Y_g^{(2)} \rangle,$$

$$(13) \quad \langle X, Y_1 Y_2 \rangle = \sum_h \langle X_h^{(1)}, Y_1 \rangle \langle X_h^{(2)}, Y_2 \rangle,$$

$$(14) \quad \langle E_i, F_j \rangle = \delta_{ij}, \quad \langle K, L \rangle = \chi(\lambda, \mu), \quad \langle E_i, L \rangle = \langle K, F_j \rangle = 0$$

for  $X, X_1, X_2 \in \tilde{U}^{+,K}(\chi)$  with  $\Delta(X) = \sum_h X_h^{(1)} X_h^{(2)}$ , and  $Y, Y_1, Y_2 \in \tilde{U}^{L,-}(\chi)$  with  $\Delta(Y) = \sum_g Y_g^{(1)} Y_g^{(2)}$  and for  $i, j \in I$  and  $\lambda, \mu \in \mathbb{Z}\Pi$ . We see

$$(15) \quad \langle \tilde{E}K, \tilde{F}L \rangle = \langle \tilde{E}, \tilde{F} \rangle \langle K, L \rangle$$

for  $\tilde{E} \in \tilde{U}^+(\chi)$  and  $\tilde{F} \in \tilde{U}^-(\chi)$ . Further, letting  $\tilde{U}^\pm(\chi) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}\Pi} \tilde{U}^\pm(\chi)_\alpha$  be the  $\mathbb{Z}_{\geq 0}\Pi$ -grading on  $\tilde{U}^\pm(\chi)$  defined in a way similar to the one on  $U^\pm(\chi)$ , we have  $\langle \tilde{U}^+(\chi), \tilde{U}^-(\chi)_- \rangle = \{0\}$  if  $\lambda \neq \mu$ . Let

$$(16) \quad \tilde{J}^+(\chi) = \{\tilde{E} \in \tilde{U}^+(\chi) \mid \langle \tilde{E}, \tilde{U}^-(\chi) \rangle = \{0\}\},$$

$$(17) \quad \tilde{J}^-(\chi) = \{\tilde{F} \in \tilde{U}^-(\chi) \mid \langle \tilde{U}^+(\chi), \tilde{F} \rangle = \{0\}\},$$

$$(18) \quad \tilde{J}(\chi) = \text{Span}_{\mathbf{k}}(\tilde{J}^+(\chi)\tilde{U}^0(\chi)\tilde{U}^-(\chi) + \tilde{U}^+(\chi)\tilde{U}^0(\chi)\tilde{J}^-(\chi)).$$

Then  $\tilde{J}(\chi)$  is the kernel of the Hopf algebra epimorphism from  $\tilde{U}(\chi)$  to  $U(\chi)$  sending the generators to the ones denoted by the same symbols.

**Theorem 1.** (Kharchenko [Kha99]) *There exist  $M \in \mathbb{N} \cup \{\infty\}$  and elements  $\hat{E}_i \in U^+(\chi)$ ,  $(1 \leq i \leq M)$  for some  $\beta_i \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}$  such that we have the  $\mathbf{k}$ -basis of  $U^+(\chi)$  formed by the elements*

$$(19) \quad \begin{cases} \hat{E}_1^{m_1} \hat{E}_2^{m_2} \cdots \hat{E}_M^{m_M} & \text{if } M \text{ is finite, that is } M \in \mathbb{N}, \\ \hat{E}_1^{m_1} \hat{E}_2^{m_2} \cdots \hat{E}_{M'}^{m_{M'}} & \text{for some } M' \in \mathbb{N} \text{ if } M = \infty \end{cases}$$

with  $0 \leq m_i \leq h_i$ , and  $h_i := \text{Max}\{n \mid [n]_{\beta_i}! \neq 0\} \in \mathbb{N} \cup \{+\infty\}$ .

Let

$$(20) \quad R_+ := \{\beta_i \mid 1 \leq i \leq M\}.$$

Note that  $|R_+| \leq M$ , that is,  $\beta_i$  and  $\beta_j$  may be the same for some  $i \neq j$ .

We say that  $\chi$  is *finite-type* if  $|R_+| < +\infty$ . See [H09] for the classification.

Note that if  $\dim U(\chi) < \infty$ , then  $\chi$  is finite-type.

**Theorem 2.** (see [HY08b, Theorems 4.8, 4.9]) Assume that  $\chi$  is finite-type. Then  $|R_+| = M$  as for (20). We write  $E_i = \widehat{E}_i$  if  $\widehat{E}_i \in U(\chi)_i$ . Then after re-choosing  $E_i$  (as in (51)), we may assume that  $E_i^{h_{\beta}^{\chi}+1} = 0$  if  $h_i < +\infty$  and that  $E_i E_j = \chi(\beta_i, \beta_j) E_j E_i \in \langle E_r | i < r < j \rangle$  for any  $i < j$ , so

$$(21) \quad \{E_{f(1)}^{m_{f(1)}} E_{f(2)}^{m_{f(2)}} \cdots E_{f(M)}^{m_{f(M)}} | 0 \leq m_i \leq h_i\}$$

is a  $\mathbb{k}$ -basis of  $U(\chi)$  for any bijective map  $f : \{1, 2, \dots, M\} \rightarrow \{1, 2, \dots, M\}$ .

*Convention.* Let  $\chi_1, \chi_2 : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \rightarrow \mathbb{k}^\times$  be two bi-characters. Let  $f_1, f_2 : U(\chi_1) \rightarrow U(\chi_2)$  be two  $\mathbb{k}$ -algebra homomorphisms. Then we write

$$(22) \quad f_1 = f_2$$

if

$$(23) \quad f_1(K L) = f_2(K L), f_1(E_i) = f_2(E_i), f_1(F_i) = f_2(F_i)$$

for all  $\lambda, \mu \in \mathbb{Z}\Pi$  and  $i \in I$ .

### 3 Heckenberger's Lusztig-type isomorphisms

Here we explain a generalization [H07] of Lusztig-type isomorphisms [L].

Assume  $\chi$  to be any bi-character. Let

$$(24) \quad \llbracket X, Y \rrbracket^+ = X Y = \chi(\lambda, \mu) Y X,$$

$$(25) \quad \llbracket X, Y \rrbracket^- = X Y = \chi(\lambda, \mu)^{-1} Y X,$$

$$(26) \quad \llbracket X, Y \rrbracket^{\vee,+} = X Y = \chi(\mu, \lambda) Y X,$$

$$(27) \quad \llbracket X, Y \rrbracket^{\vee,-} = X Y = \chi(\mu, \lambda)^{-1} Y X$$

for  $X \in U(\chi)$  and  $Y \in U(\chi)$  with  $\lambda, \mu \in \mathbb{Z}\Pi$ . Let  $i, j \in I$  be such that  $i \neq j$ . Let

$$(28) \quad \begin{aligned} E_j^+ &= E_j, \quad E_j^- = E_j, \\ E_{j+m}^+ &= \llbracket E_i, E_{j+(m-1)}^+ \rrbracket, \quad E_{j+m}^- = \llbracket E_i, E_{j+(m-1)}^- \rrbracket^{\vee,-}, \\ F_{j+m}^+ &= \llbracket F_i, F_{j+(m-1)}^+ \rrbracket^{\vee,+}, \quad F_{j+m}^- = \llbracket F_i, F_{j+(m-1)}^- \rrbracket^- \end{aligned}$$

for  $m \in \mathbb{N}$ . For  $m \in \mathbb{Z}_{\geq 0}$ , we have

$$(29) \quad \begin{aligned} [m]_{(\alpha_i, \alpha_i)}! \prod_{s=1}^m (1 - \chi(\alpha_i, \alpha_i)^{s-1} \chi(\alpha_i, \alpha_j) \chi(\alpha_j, \alpha_i)) &\neq 0 \\ \iff E_{j+m}^+ &\neq 0 \iff E_{j+m}^- \neq 0 \iff F_{j+m}^+ \neq 0 \iff F_{j+m}^- \neq 0 \\ &\iff \alpha_j + m\alpha_i \in R_+. \end{aligned}$$

We also have

$$(30) \quad [E_{j+m_i}^+, F_{j+m_i}^+] = (\chi(\alpha_i, \alpha_i)^{m-1} \chi(\alpha_i, \alpha_j) \chi(\alpha_j, \alpha_i))^m [E_{j+m_i}^-, F_{j+m_i}^-]$$

$$= (-1)^m ([m]_{(-i, i)}!) \prod_{s=1}^m (1 - \chi(\alpha_i, \alpha_i)^{s-1} \chi(\alpha_i, \alpha_j) \chi(\alpha_j, \alpha_i)) (K_{j+m_i} - L_{j+m_i}).$$

**Theorem 3.** ([H07]) *Let  $i \in I$ . Assume that for all  $j \in I \setminus \{i\}$ , there exist  $m_{ij} \in \mathbb{Z}_{\geq 0}$  such that  $E_{j+m_{ij}i}^+ \neq 0$  and  $E_{j+(m_{ij}+1)i}^+ = 0$ .*

(1) *There exist a bi-character  $r_i(\chi) : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \rightarrow \mathbb{k}^\times$  and  $\mathbb{k}$ -algebra isomorphisms*

$$(31) \quad T_i = T_i^+ : U(r_i(\chi)) \rightarrow U(\chi), \quad T_i^- : U(r_i(\chi)) \rightarrow U(\chi)$$

*such that*

$$(32) \quad T_i^\pm(K_{-i}) = K_{-i}, \quad T_i^\pm(L_{-i}) = L_{-i},$$

$$(33) \quad T_i^\pm(K_j) = K_{j+m_{ij}i}, \quad T_i^\pm(L_j) = L_{j+m_{ij}i},$$

$$(34) \quad T_i(E_i) = F_i L_{-i}, \quad T_i(F_i) = K_{-i} E_i,$$

$$(35) \quad T_i^-(E_i) = K_{-i} F_i, \quad T_i^-(F_i) = E_i L_{-i},$$

$$(36) \quad T_i^\pm(E_j) = E_{j+m_{ij}i}^\pm, \quad T_i^\pm(F_j) = F_{j+m_{ij}i}^\pm,$$

*where  $j \in I \setminus \{i\}$ .*

(2)  $r_i(r_i(\chi))$  exists in the same way as above with  $r_i(\chi)$  in place of  $\chi$ . Further  $r_i(r_i(\chi)) = \chi$ ,  $m_{ij}^{r_i(\cdot)} = m_{ij}$  for all  $j \in I \setminus \{i\}$ .

(3) Let  $T_i : U(r_i(\chi)) \rightarrow U(\chi)$  be as in (31). Let  $T_i^- : U(\chi) \rightarrow U(r_i(\chi))$  be the one as in (31) defined with  $r_i(\chi)$  in place of  $\chi$ . Then  $T_i^- T_i = \text{id}_{U(r_i(\chi))}$  and  $T_i T_i^- = \text{id}_{U(\chi)}$ .

(4) Define the  $\mathbb{Z}$ -module isomorphism  $\sigma_i^{r_i(\cdot)} = \sigma_i : \mathbb{Z}\Pi \rightarrow \mathbb{Z}\Pi$  by  $T_i^\pm(U(r_i(\chi))) = U(\chi)_{i(\cdot)}$  for all  $\lambda \in \mathbb{Z}\Pi$ . Then

$$(37) \quad \sigma_i^{r_i(\cdot)} = \sigma_i, \quad \sigma_i \sigma_i^{r_i(\cdot)} = \text{id}_{\mathbb{Z}\Pi}$$

and

$$(38) \quad \sigma_i^{r_i(\cdot)}(R_+^{r_i(\cdot)} \setminus \{\alpha_i\}) = R_+ \setminus \{\alpha_i\}, \quad \sigma_i^{r_i(\cdot)}(\alpha_i) = -\alpha_i.$$

**Theorem 4.** ([H07]) *Assume  $\chi$  to be finite-type. Let  $i, j \in I$  to be such that  $i \neq j$ . Let  $M = |R_+ \cap (\mathbb{Z}_{\geq 0}\alpha_i - \mathbb{Z}_{\geq 0}\alpha_j)|$ . For  $n \in \{1, 2, \dots, M\}$ , define two bi-characters  $\chi_n, \chi'_n$ , two  $\mathbb{Z}$ -module automorphism  $\bar{\sigma}_n, \bar{\sigma}'_n$  of  $\mathbb{Z}\Pi$  and two  $\mathbb{k}$ -algebra*

isomorphisms  $\bar{T}_n : U(\chi_n) \rightarrow U(\chi)$ ,  $\bar{T}'_n : U(\chi'_n) \rightarrow U(\chi)$  in the way that  $\chi_1 = \chi'_1 = \chi$ ,  $\bar{\sigma}_1 = \bar{\sigma}'_1 = \text{id}_{\mathbb{Z}\Pi}$ ,  $\bar{T}_1 = \bar{T}'_1 = \text{id}_{U(\chi)}$ , and

$$(39) \quad \chi_{2n} = r_i(\chi_{2n-1}), \chi_{2n+1} = r_j(\chi_{2n}), \chi'_{2n} = r_j(\chi'_{2n-1}), \chi'_{2n+1} = r_i(\chi'_{2n}),$$

$$(40) \quad \bar{\sigma}_{2n} = \bar{\sigma}_{2n-1}\sigma_i^{2n}, \bar{\sigma}_{2n+1} = \bar{\sigma}_{2n}\sigma_j^{2n+1}, \bar{\sigma}'_{2n} = \bar{\sigma}'_{2n-1}\sigma_j^{2n}, \bar{\sigma}'_{2n+1} = \bar{\sigma}'_{2n}\sigma_i^{2n+1},$$

$$(41) \quad \bar{T}_{2n} = \bar{T}_{2n-1}T_i, \bar{T}_{2n+1} = \bar{T}_{2n}T_j, \bar{T}'_{2n} = \bar{T}'_{2n-1}T_j, \bar{T}'_{2n+1} = \bar{T}'_{2n}T_i.$$

Then we have

$$(42) \quad \chi_M = \chi'_M,$$

$$(43) \quad \bar{\sigma}_M = \bar{\sigma}'_M$$

and

$$(44) \quad \bar{T}_M = \bar{T}'_M.$$

## 4 Longest elements of Weyl groupoids

In this section we always assume  $\chi$  to be finite-type, and refer to [CH08] for categorical definitions of Weyl groupoids.

*Convention.* For a category  $\mathcal{C}$ , we denote the product of the morphisms by  $\cdot$ . That is, for two morphism  $f_1 \in \text{Mor}(a_1, b_1)$  and  $f_2 \in \text{Mor}(a_2, b_2)$  with  $a_1, b_1, a_2$  and  $b_2 \in \text{Ob}(\mathcal{C})$ , we denote their product by

$$(45) \quad f_1 \cdot f_2 \quad \text{if } b_2 = a_1.$$

Set

$$(46) \quad \mathcal{C}(\chi) = \{\chi\} \cup \bigcup_{n=1}^{\infty} \{r_{i_1} \cdots r_{i_n}(\chi) \mid i_1, \dots, i_n \in I\}.$$

Let  $W = W(\chi)$  be the category with  $\text{Ob}(W) = \mathcal{C}(\chi)$  and generated by the maps  $\sigma_i' \in \text{Mor}_W(\chi', r_i(\chi'))$  with  $\chi' \in \text{Ob}(W)$  and  $i \in I$ . Let  $\mathcal{W} = \mathcal{W}(\chi)$  be the (abstract) category with  $\text{Ob}(\mathcal{W}) = \mathcal{C}(\chi)$  defined by generators  $s_i' \in \text{Mor}_{\mathcal{W}}(\chi', r_i(\chi'))$  with  $\chi' \in \text{Ob}(W)$  and  $i \in I$  and relations

$$(47) \quad s_i' \cdot s_i^{r_i(\chi')} = 1_{r_i(\chi')},$$

$$(48) \quad s_i' s_j^{r_i(\cdot)} s_i^{r_j r_i(\cdot)} = s_j' s_i^{r_j(\cdot)} s_j^{r_i r_j(\cdot)}$$

(both sides are composed of  $|R_+ \cap (\mathbb{Z}\alpha_i - \mathbb{Z}\alpha_j)|$ -factors).

We call  $\mathcal{W}$  the *Weyl groupoid*. Define the morphism  $\phi: \mathcal{W} \rightarrow W$  by  $\phi(s_i') = \sigma_i'$ . Then  $\phi$  is bijective, see [HY08a, Theorem 1]. Let  $\ell(1') = 0$  for  $\chi' \in \mathcal{C}(\chi)$ . Let  $\ell(s_i') = 1$ . For  $w \in \text{Mor}_{\mathcal{W}}(\chi_1, \chi_2)$ , let  $\ell(w)$  be the least number  $\ell(w') + \ell(w'')$  with  $w = w' w''$  for some  $\chi_3 \in \mathcal{C}(\chi)$ , and some  $w' \in \text{Mor}_{\mathcal{W}}(\chi_3, \chi_2)$ , some  $w'' \in \text{Mor}_{\mathcal{W}}(\chi_1, \chi_3)$ . By [HY08a, Lemma 8(iii)], we have

$$(49) \quad \ell(w) = |\{\alpha \in R_+^1 | \phi(w)(\alpha) \in -R_+^2\}|.$$

Moreover for each  $\chi_1 \in \mathcal{C}(\chi)$ , there exists unique  $\chi_2 \in \mathcal{C}(\chi)$  and  ${}^1w_0 \in \text{Mor}_{\mathcal{W}}(\chi_2, \chi_1)$  such that  $\phi({}^1w_0)(R_+^2) = -R_+^1$ . We call  ${}^1w_0$  the *longest element* since  $\ell({}^1w_0) = \ell(w')$  for any  $w' \in \text{Mor}_{\mathcal{W}}(\chi_3, \chi_4)$  for any  $\chi_3, \chi_4 \in \mathcal{C}(\chi)$ .

Let  $\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}(\chi)$  be the (abstract) category with  $\text{Ob}(\widetilde{\mathcal{W}}) = \mathcal{C}(\chi)$  defined by generators  $\widetilde{s}_i' \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi', r_i(\chi'))$  with  $\chi' \in \text{Ob}(W)$  and  $i \in I$  and relations

$$(50) \quad \widetilde{s}_i' \widetilde{s}_j^{r_i(\cdot)} \widetilde{s}_i^{r_j r_i(\cdot)} = \widetilde{s}_j' \widetilde{s}_i^{r_j(\cdot)} \widetilde{s}_j^{r_i r_j(\cdot)}$$

(both sides are composed of  $|R_+ \cap (\mathbb{Z}\alpha_i - \mathbb{Z}\alpha_j)|$ -factors).

Let  $\widetilde{1}' \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi', \chi')$  denote the identity morphism. Define the morphism  $\widetilde{\phi}: \widetilde{\mathcal{W}} \rightarrow \mathcal{W}$  by  $\widetilde{\phi}(\widetilde{s}_i') = s_i'$ .

Let  $\widetilde{\ell}(\widetilde{1}') = 0$  for  $\chi' \in \mathcal{C}(\chi)$ . Let  $\widetilde{\ell}(\widetilde{s}_i') = 1$ . For  $\widetilde{w} \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_2)$ , let  $\widetilde{\ell}(\widetilde{w})$  be the least number  $\widetilde{\ell}(\widetilde{w}') + \widetilde{\ell}(\widetilde{w}'')$  with  $\widetilde{w} = \widetilde{w}' \widetilde{w}''$  for some  $\chi_3 \in \mathcal{C}(\chi)$ , and some  $\widetilde{w}' \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_3, \chi_2)$ , some  $\widetilde{w}'' \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_3)$ .

**Theorem 5.** ([HY08a, Theorem 5, Corollary 6]) *Let  $\chi_1, \chi_2 \in \mathcal{C}(\chi)$ . For  $w \in \text{Mor}_{\mathcal{W}}(\chi_1, \chi_2)$  and  $\widetilde{w}_1, \widetilde{w}_2 \in \widetilde{w} \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_2)$  with  $\widetilde{\phi}(\widetilde{w}_1) = \widetilde{\phi}(\widetilde{w}_2) = w$  and  $\ell(w) = \widetilde{\ell}(\widetilde{w}_1) = \widetilde{\ell}(\widetilde{w}_2)$ , we have  $\widetilde{w}_1 = \widetilde{w}_2$ . Further, if  $\widetilde{w} \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_2)$  is such that  $\widetilde{\ell}(\widetilde{w}) > \ell(\widetilde{\phi}(\widetilde{w}))$ , then  $\widetilde{w} = \widetilde{w}' \widetilde{s}_i^{r_i(\cdot)} \widetilde{s}_i^{r_i(\cdot)} \widetilde{w}''$  for some  $i \in I$ ,  $\widetilde{w}' \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_3)$  and  $\widetilde{w}'' \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_3, \chi_2)$  with  $\widetilde{\ell}(\widetilde{w}') + \widetilde{\ell}(\widetilde{w}'') = \widetilde{\ell}(\widetilde{w}) - 2$ .*

Assume  $w_0$  to be  $s_{j_1}^1 s_{j_2}^2 \dots s_{j_M}^M$ , where  $M = |R_+|$ ,  $r_1(\chi_1) = \chi$ , and  $r_j(\chi_j) = \chi_{j-1}$ . Let  $\bar{T}_1 = \text{id}_{U(\cdot)}$ . For  $2 \leq n \leq M$ , define the  $\mathbb{k}$ -algebra isomorphism  $\bar{T}_n: U(\chi_{n-1}) \rightarrow U(\chi)$  by  $\bar{T}_n = \bar{T}_{n-1} T_{j_{n-1}}$ . Then as for  $E_i$  of Theorem 2, we may put

$$(51) \quad E_i = \bar{T}_i(E_{j_i})$$

for  $1 \leq j \leq M$ .



## 5 Shapovalov determinants

Let  $\chi$  be a bi-character. We define the *Shapovalov matrix*  $\text{Sh}$  in the natural way for each  $\alpha \in \mathbb{Z}_{\geq 0}\Pi$ . More precisely,  $\text{Sh}$  is a  $\dim U^+(\chi) \times \dim U^+(\chi)$ -matrix whose components are elements of  $U^0(\chi)$ . Let  $\rho : \mathbb{Z}\Pi \rightarrow \mathbb{k}^\times$  be the (abelian) group homomorphism defined by  $\rho(\alpha_i) = \chi(\alpha_i, \alpha_i)$ . We use the *Kostant partition function*  $P(\alpha, \beta, t) := \dim E^t U^+(\chi)_{-\alpha - \beta}$ , where we define  $P(\alpha, \beta, t) = 0$  in case  $\alpha - t\beta \notin \mathbb{Z}_{\geq 0}\Pi$ .

**Theorem 6.** ([HY08b, Theorem 7.3]) *Let  $\chi$  be finite-type. Assume that  $\chi(\beta, \beta) \neq 1$  for all  $\beta \in R_+$ . Then for  $\alpha \in \mathbb{Z}_{\geq 0}\Pi$ , we have*

$$\det \text{Sh} = c \prod_{\beta \in R_+} \prod_{t=1}^{h_\beta^\chi} (\rho(\beta) K^{-\chi(\beta, \beta)t} L)^{P^\chi(\alpha, t)}$$

for some  $c \in \mathbb{k}^\times$ .

As stated below, for  $U(\chi)$  which is the (ordinary or small) quantum group of a finite dimensional Lie algebra  $\mathfrak{g}$ , we have the generalization of (1) the one [dDK90] for  $q \in \mathbb{C}^\times$  which is not a root of unity, and (2) the one [KL97] for  $q \in \mathbb{C}^\times$  which is a primitive  $p$ -th root of unity for some prime number  $p$ .

**Corollary 7.** *Let  $\mathfrak{g}$  be a finite dimensional simple Lie algebra of type A-G or a finite dimensional simple Lie superalgebra of type A-G. Then the Shapovalov determinant of the quantum group  $U_q(\mathfrak{g})$  when  $q$  is not root of unity or the small quantum group  $u_q(\mathfrak{g})$  when  $q$  is a primitive  $r$ -th root of unity for some positive integer  $r \geq 2$  is given by*

$$c \prod_{\beta \in R_+} \prod_{t=1}^{h_\beta^\chi} (q^{2(\beta, \beta)t} K^{-q(\beta, \beta)t} K^{-1})^{P^\chi(\alpha, t)}$$

for some  $c \in \mathbb{C}^\times$ .

We even recover the original ones due to Shapovalov [Sha72], and Kac [Kac77] (super cases):

**Corollary 8.** *Let  $\mathfrak{g}$  be as above. Then the Shapovalov determinant of the enveloping algebra  $U(\mathfrak{g})$  is given by*

$$c \prod_{\beta \in R_+} \prod_{t=1}^{\infty} (H + (\rho, \beta) - \frac{(\beta, \beta)t}{2})^{P^\chi(\alpha, t)}$$

for some  $c \in \mathbb{C}^\times$ .

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## References

- [AS98] N. Andruskiewitsch and H.-J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order  $p^3$ , *J. Algebra* 209 (1998), 658-691.
- [AS02] ———, Pointed Hopf algebras, *New Directions in Hopf Algebras*, MSRI Publications, vol. 43, Cambridge University Press, 2002.
- [CH08] M. Cuntz, I. Heckenberger, *Weyl groupoids and at most three objects* *J. Pure Appl. Algebra* 213 (2009) 1112-1128
- [Hec06] I. Heckenberger, The Weyl groupoid of a Nichols algebra of diagonal type, *Invent. Math.* 164 (2006), 175-188.
- [H09] ———, *Classification of arithmetic root systems*, *Adv. Math.* 220 (2009) 59-124.
- [H07] ———, *Lusztig isomorphisms for Drinfel'd doubles of Nichols algebras of diagonal type*, Preprint arXiv:0710.4521v1 (2007)
- [HY08a] I. Heckenberger, H. Yamane, *A generalization of Coxeter groups, root systems, and Matsumoto's theorem*, *Math. Z.* 259 (2008) 255-276.
- [HY08b] ———, *Drinfel'd doubles and Shapovalov determinants*, preprint arXiv:0810.1621 (2008)
- [Kac77] V.G. Kac, Representation of classical Lie superalgebras, *Lecture Notes in Physics*, vol. 676, pp. 597-626, Springer-Verlag, 1977.
- [Kha99] V. Kharchenko, *A quantum analogue of the Poincaré-Birkhoff-Witt theorem*, *Algebra and Logic* 38 (1999), 259-276
- [dDK90] C. de Concini, V.G. Kac, *Representations of quantum groups at roots of 1*, *Progress in Math.* vol. 92, Birkhäuser, 1990, pp. 471-506
- [KL97] S. Kummer, G. Letzter, *Shapovalov determinant for restricted and quantized restricted enveloping algebras*, *Pacific. J. Math.* 179 (1997) no. 1, 123-161
- [L] G. Lusztig, *Introduction to quantum groups*, Birkhäuser, Boston, MA 1993

[Sha72] N.N. Shapovalov, On a bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra, *Funct. Anal. Appl* 6 (1972), 307-312.

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